Statistics Class Problem Solutions (Problems 1-3)

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Suppose the return distributions for two risky assets are as follows:

State	p_s	$r_{a,s}$	$r_{b,s}$
1	1/3	-3%	36%
2	1/3	9%	-12%
3	1/3	21%	12%

1. Calculate the expected returns for assets a and b.

$$E(r_a) = \sum_{s=1}^{n} p_s r_{a,s} = (1/3)(-3\%) + (1/3)(9\%)(1/3)(21\%) = 9\%$$
$$E(r_b) = \sum_{s=1}^{n} p_s r_{b,s} = (1/3)(36\%) + (1/3)(-12\%)(1/3)(12\%) = 12\%$$

2. Calculate the variances and standard deviations for assets a and b.

$$\sigma_a^2 = \sum_{s=1}^{n} p_s (r_{a,s} - E(r_a))^2 = (1/3)(-3\% - 9\%)^2 + (1/3)(9\% - 9\%)^2 + (1/3)(21\% - 9\%)^2 = .96\%$$
$$\sigma_a = \sqrt{.96\%} = 9.8\%$$
$$\sigma_b^2 = \sum_{s=1}^{n} p_s (r_{a,s} - E(r_a))^2 = (1/3)(36\% - 12\%)^2 + (1/3)(-12\% - 12\%)^2 + (1/3)(12\% - 12\%)^2 = 3.84\%$$
$$\sigma_b = \sqrt{3.84\%} = 19.6\%$$

3. Calculate the covariance and correlation between assets a and b.

$$\sigma_{ab} = \sum_{s=1}^{n} p_s (r_{a,s} - E(r_a))(r_{b,s} - E(r_b))$$
$$= (1/3)(-12\%)(24\%) + (1/3)(21\%)(-24\%) + (1/3)(12\%)(0) = -.96\%$$
$$\rho_{ab} = \frac{\sigma_{ab}}{\sigma_a \sigma_b} = \frac{.96\%}{(9.8\%)(19.6\%)} = -.50$$

4. Calculate the expected return and standard deviation for an equally weighted portfolio consisting of asset a and b.

$$E(r_p) = \sum_{s=1}^{n} w_i E(r_i) = .5(9\%) + .5(12\%) = 10.5\%$$
$$\sigma_p = \sqrt{w_a^2 \sigma_a^2 + w_b^2 \sigma_b^2 + 2 w_a w_b \sigma_{ab}} = \sqrt{.25(.96\%) + .25(3.84\%) + 2(.5)(.5)(-.96\%)} = 8.49\%$$

5. Determine the least risky combination of assets a and b and calculate the expected return and standard deviation for such a portfolio.

   We can rewrite $\sigma_p^2 = w_a^2\sigma_a^2 + w_b^2\sigma_b^2 + 2w_a w_b \sigma_{ab} = w_a^2\sigma_a^2 + (1-w_a)^2\sigma_b^2 + 2w_a(1-w_a)\sigma_{ab}$. Our task is to select w_a such that σ_p^2 is minimized. We do this by differentiating σ_p^2 with respect to w_a, setting the resulting equation to 0, and solving for w_a. Thus, the following expression obtains:

$$w_a = \frac{\sigma_b^2 - \sigma_{ab}}{\sigma_a^2 + \sigma_b^2 - 2\sigma_{ab}} = \frac{3.84\% + .96\%}{.96\% + 3.84\% + 2(.96\%)} = 5/7$$

$$w_b = 1 - w_a = 2/7$$

$$E(r_p) = \sum_{s=1}^{n} w_i E(r_i) = (5/7)(9\%) + (2/7)(12\%) = 9.86\%$$

$$\sigma_p = \sqrt{w_a^2\sigma_a^2 + w_b^2\sigma_b^2 + 2w_a w_b \sigma_{ab}} = \sqrt{(25/49)(.96\%) + (4/49)(3.84\%) + 2(5/7)(2/7)(-.96\%)} = 6.41\%$$

   Note that the coefficient of variation, given by the ratio $\sigma_p/E(r_p)$, is .81 for the equally weighted portfolio compared with .65 for the least risky portfolio consisting of assets a and b; i.e., in order to earn a slightly higher expected return from the equally weighted portfolio, the investor must be willing to take on proportionately greater risk. Whether or not an investor is willing to bear this extra risk in order to earn the extra return depends upon the investor's degree of risk aversion. Graphically, both the least risky portfolio and the equally weighted portfolio are mean-variance efficient.