CHAPTER 4: Portfolio Theory

Though one may be overpowered, two can defend themselves. A cord of three strands is not quickly broken.

-- Ecclesiastes 4:12 (NIV)

But divide your investments among many places, for you do not know what risks might lie ahead.

-- Ecclesiastes 11:2 (NLT)

Mean Variance Efficiency

- A portfolio is *mean variance efficient* (MVE) if there is no other portfolio which, for a given level of variance, has a higher expected return.
- Equivalently, a portfolio is MVE if there is no other portfolio which, for a given level of expected return, has lower variance.

$E(U(W)) = f(E(W), \sigma^2_W)$

Expected Return and Risk Calculations

We calculate expected returns, standard deviations, and covariances on individual securities in the following manner:

$$E(r_i) = \sum_{s=1}^{n} p_s r_{i,s} \quad (1)$$

$$\sigma_i = \sqrt{\sum_{s=1}^{n} p_s (r_{i,s} - E(r_i))^2} \quad (2)$$

$$\sigma_{i,j} = \sum_{s=1}^{n} p_s (r_{i,s} - E(r_i))(r_{j,s} - E(r_j)) \quad (3)$$

$\rho_{ij} = \sigma_{ij} / \sigma_i \sigma_j$

Portfolio Return and Risk Calculations

Portfolio expected returns and standard deviations are calculated as follows:

$$E(r_p) = \sum_{i=1}^{n} w_i E(r_i) \tag{4}$$

$$\sigma_p = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j}} \tag{5}$$

where w_i is the proportion of total investment in asset i, σ_i^2 is the variance of asset i, and σ_{ij} is the covariance between assets i and j.

Minimum Risk Portfolios (2 assets)

For a two-asset portfolio, portfolio variance is written as:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}. \qquad (6)$$

Since we wish to find the least risky combination of assets 1 and 2, we differentiate equation (6) with respect to w_1, set the result equal to zero, and solve for w_1.

Minimum Risk Portfolios (2 assets)

The problem is easier if we first substitute $w_2 = (1- w_1)$ on the right hand side of equation (6) and simplify:

$$\begin{aligned}
\sigma_p^2 &= w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\sigma_{12} \\
&= w_1^2\sigma_1^2 + (1-w_1)(1-w_1)\sigma_2^2 + 2w_1\sigma_{12} - 2w_1^2\sigma_{12} \\
&= w_1^2\sigma_1^2 + \sigma_2^2 + w_1^2\sigma_2^2 - 2w_1\sigma_2^2 + 2w_1\sigma_{12} - 2w_1^2\sigma_{12} \\
&= w_1^2(\sigma_1^2 + \sigma_2^2) + 2w_1(\sigma_{12} - \sigma_2^2) + \sigma_2^2 - 2w_1^2\sigma_{12}. \qquad (7)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d\sigma_p^2}{dw_1} &= 2w_1(\sigma_1^2 + \sigma_2^2) + 2(\sigma_{12} - \sigma_2^2) - 4w_1\sigma_{12} \\
&= w_1(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}) + \sigma_{12} - \sigma_2^2 = 0. \qquad (8)
\end{aligned}$$

Minimum Risk Portfolios (2 assets)

Solving equation (8) for w_1 yields:

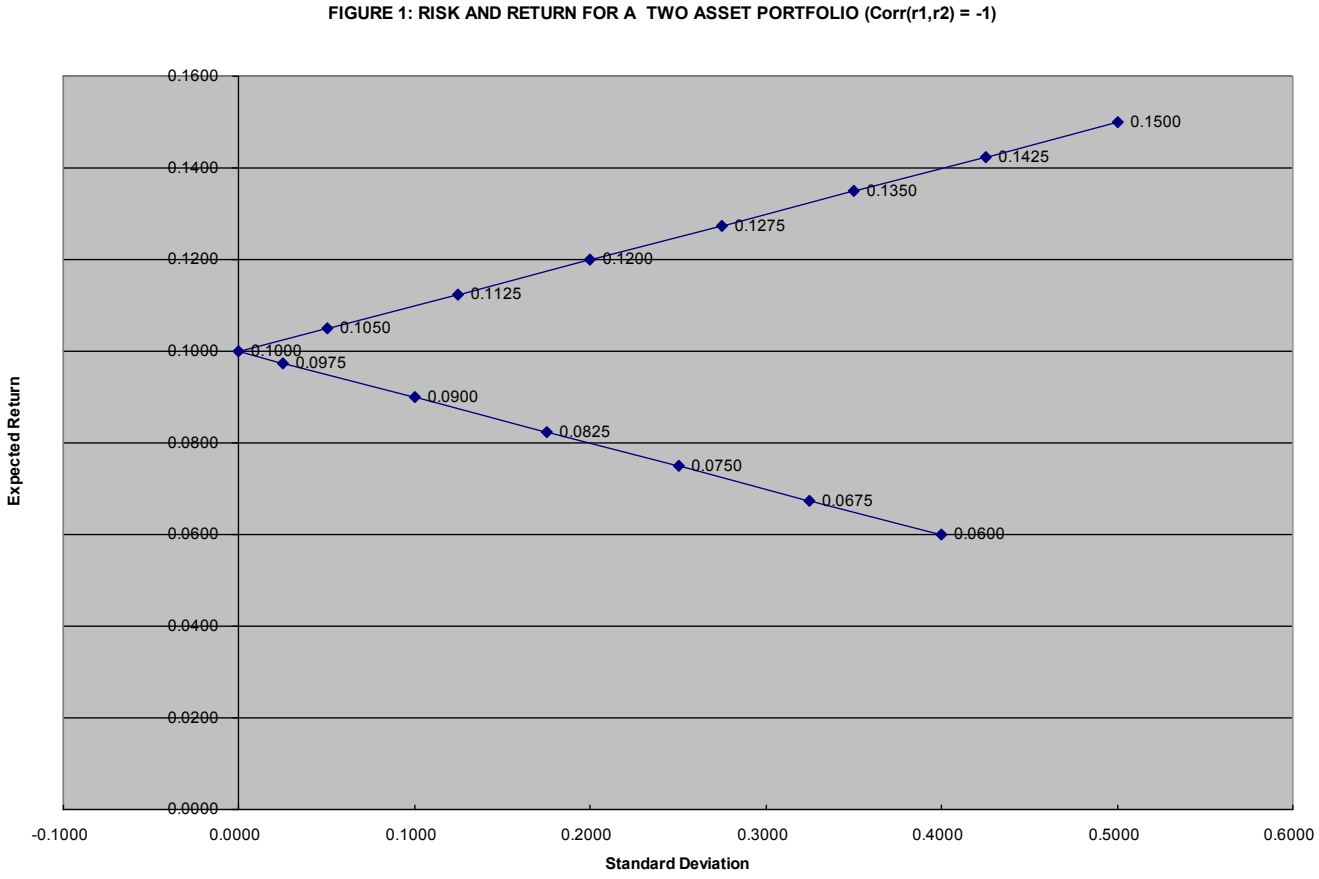
$$w_1 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}. \qquad (9)$$

- By weighting 2-asset portfolios according to the ratio given by equation (9), we are guaranteed a portfolio combination that *minimizes* total portfolio risk.
- The expected utility rule implies that we will select portfolios which maximize expected return for a given level of risk.
- Therefore, we limit our selection to only those portfolios which feature this property of "mean-variance efficiency".
  - The minimum risk portfolio given by equation (9) represents the "end point" of the set of efficient portfolios.

Portfolio Efficiency when $\rho = -1$

Risks and Returns of Alternative Portfolio Strategies					
E(Ri)	9.00%	12.00%			
Var(Ri)	1.00%	4.00%			
Std(Ri)	10.00%	20.00%			
Cov(Ra,Rb)	-0.0200				
Corr(Ra,Rb)	-1.0000				
Minimum Var.	Xa	Xb	E(Rp)	Var(Rp)	Std(Rp)
Portfolio	66.67%	33.33%	10.00%	0.00%	0.00%
Other	**-100.00%**	**200.00%**	**15.00%**	**25.01%**	**50.01%**
Portfolios	**-75.00%**	**175.00%**	**14.25%**	**18.07%**	**42.51%**
	-50.00%	**150.00%**	**13.50%**	**12.26%**	**35.01%**
	-25.00%	**125.00%**	**12.75%**	**7.57%**	**27.51%**
	0.00%	**100.00%**	**12.00%**	**4.00%**	**20.00%**
	25.00%	**75.00%**	**11.25%**	**1.56%**	**12.50%**
	50.00%	**50.00%**	**10.50%**	**0.25%**	**5.00%**
	66.67%	**33.33%**	**10.00%**	**0.00%**	**0.00%**
	75.00%	25.00%	9.75%	0.06%	2.50%
	100.00%	0.00%	9.00%	1.00%	10.00%
	125.00%	-25.00%	8.25%	3.06%	17.50%
	150.00%	-50.00%	7.50%	6.25%	25.01%
	175.00%	-75.00%	6.75%	10.57%	32.51%
	200.00%	-100.00%	6.00%	16.01%	40.01%

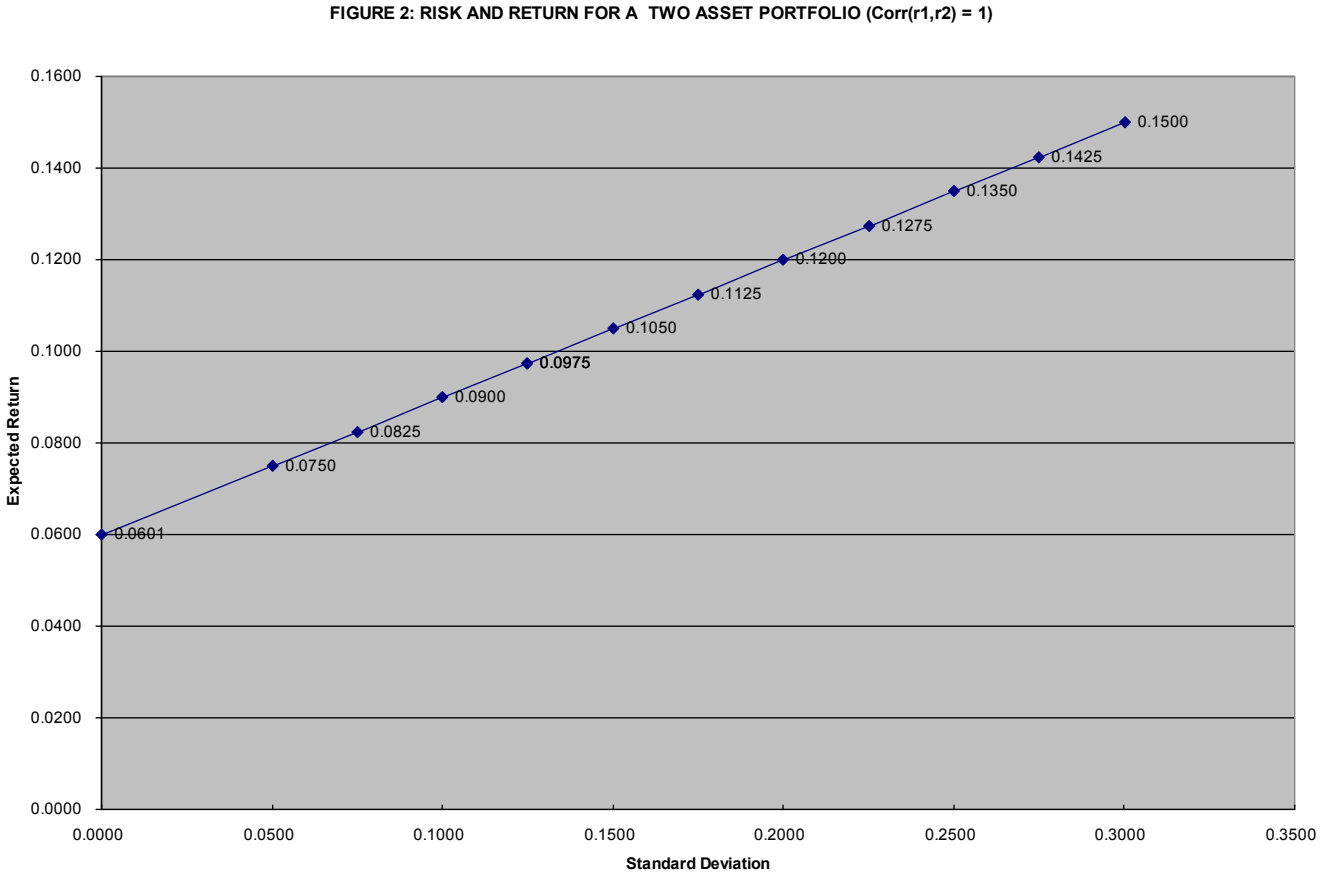
Portfolio Efficiency when $\rho = -1$



Portfolio Efficiency when $\rho = 1$

Risks and Returns of Alternative Portfolio Strategies					
E(Ri)	9.00%	12.00%			
Var(Ri)	1.00%	4.00%			
Std(Ri)	10.00%	20.00%			
Cov(Ra,Rb)	0.0200				
Corr(Ra,Rb)	1.0000				
Minimum Var.	Xa	Xb	E(Rp)	Var(Rp)	Std(Rp)
Portfolio	199.75%	-99.75%	6.01%	0.00%	0.00%
Other	**-100.00%**	**200.00%**	**15.00%**	**9.01%**	**30.01%**
Portfolios	**-75.00%**	**175.00%**	**14.25%**	**7.57%**	**27.51%**
	-50.00%	**150.00%**	**13.50%**	**6.25%**	**25.01%**
	-25.00%	**125.00%**	**12.75%**	**5.07%**	**22.51%**
	0.00%	**100.00%**	**12.00%**	**4.00%**	**20.00%**
	25.00%	**75.00%**	**11.25%**	**3.06%**	**17.50%**
	50.00%	**50.00%**	**10.50%**	**2.25%**	**15.00%**
	75.00%	**25.00%**	**9.75%**	**1.56%**	**12.50%**
	100.00%	**0.00%**	**9.75%**	**1.56%**	**12.50%**
	125.00%	**-25.00%**	**9.00%**	**1.00%**	**10.00%**
	150.00%	**-50.00%**	**8.25%**	**0.56%**	**7.51%**
	175.00%	**-75.00%**	**7.50%**	**0.25%**	**5.01%**
	199.75%	**-99.75%**	**6.01%**	**0.00%**	**0.00%**

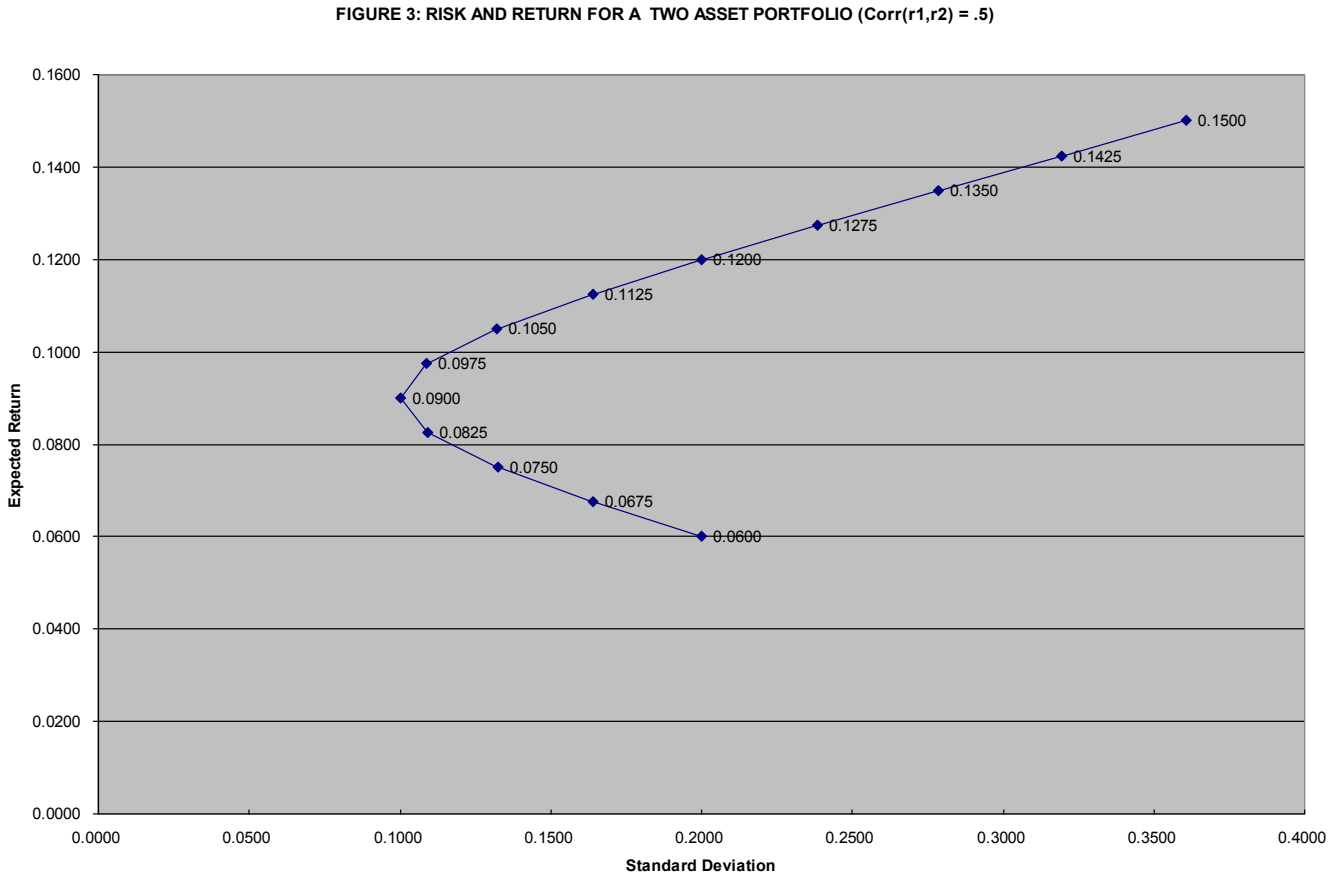
Portfolio Efficiency when $\rho = 1$



Portfolio Efficiency when $\rho = .5$

Risks and Returns of Alternative Portfolio Strategies					
E(Ri)	9.00%	12.00%			
Var(Ri)	1.00%	4.00%			
Std(Ri)	10.00%	20.00%			
Cov(Ra,Rb)	0.0100				
Corr(Ra,Rb)	0.5000				
Minimum Var.	Xa	Xb	E(Rp)	Var(Rp)	Std(Rp)
Portfolio	100.00%	0.00%	9.00%	1.00%	10.00%
Other	**-100.00%**	**200.00%**	**15.00%**	**13.01%**	**36.07%**
Portfolios	**-75.00%**	**175.00%**	**14.25%**	**10.19%**	**31.93%**
	-50.00%	**150.00%**	**13.50%**	**7.75%**	**27.85%**
	-25.00%	**125.00%**	**12.75%**	**5.69%**	**23.85%**
	0.00%	**100.00%**	**12.00%**	**4.00%**	**20.00%**
	25.00%	**75.00%**	**11.25%**	**2.69%**	**16.40%**
	50.00%	**50.00%**	**10.50%**	**1.75%**	**13.23%**
	75.00%	**25.00%**	**9.75%**	**1.19%**	**10.90%**
	100.00%	**0.00%**	**9.00%**	**1.00%**	**10.00%**
	125.00%	-25.00%	8.25%	1.19%	10.90%
	150.00%	-50.00%	7.50%	1.75%	13.23%
	175.00%	-75.00%	6.75%	2.69%	16.40%
	200.00%	-100.00%	6.00%	4.00%	20.01%

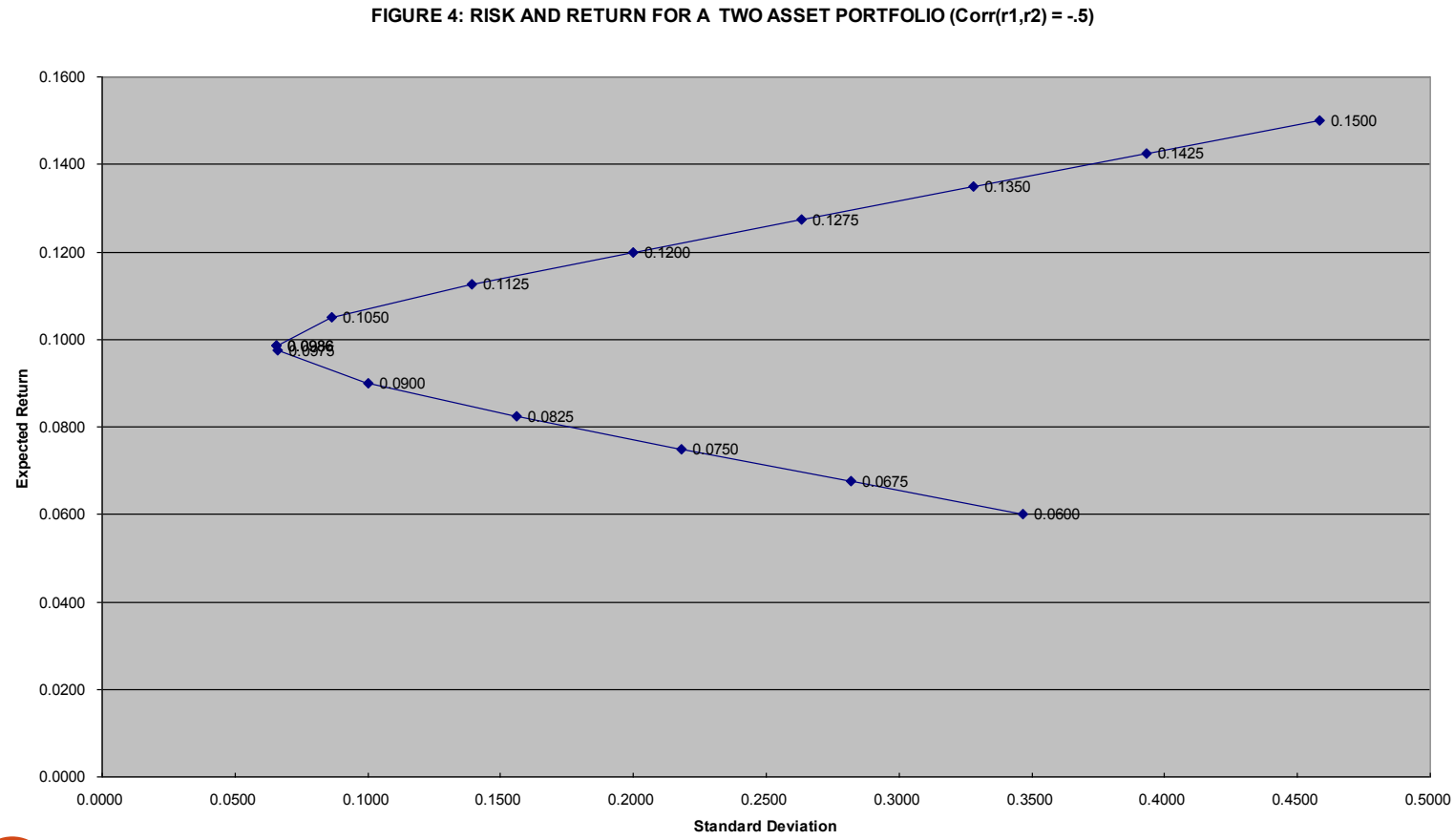
Portfolio Efficiency when $\rho = .5$



Portfolio Efficiency when $\rho = -.5$

Risks and Returns of Alternative Portfolio Strategies					
E(Ri)	9.00%	12.00%			
Var(Ri)	1.00%	4.00%			
Std(Ri)	10.00%	20.00%			
Cov(Ra,Rb)	-0.0100				
Corr(Ra,Rb)	-0.5000				
Minimum Var.	Xa	Xb	E(Rp)	Var(Rp)	Std(Rp)
Portfolio	71.43%	28.57%	9.86%	0.43%	6.55%
Other	**-100.00%**	**200.00%**	**15.00%**	**21.01%**	**45.83%**
Portfolios	**-75.00%**	**175.00%**	**14.25%**	**15.44%**	**39.30%**
	-50.00%	**150.00%**	**13.50%**	**10.75%**	**32.79%**
	-25.00%	**125.00%**	**12.75%**	**6.94%**	**26.34%**
	0.00%	**100.00%**	**12.00%**	**4.00%**	**20.00%**
	25.00%	**75.00%**	**11.25%**	**1.94%**	**13.92%**
	50.00%	**50.00%**	**10.50%**	**0.75%**	**8.66%**
	71.43%	**28.57%**	**9.86%**	**0.43%**	**6.55%**
	75.00%	25.00%	9.75%	0.44%	6.62%
	100.00%	0.00%	9.00%	1.00%	10.00%
	125.00%	-25.00%	8.25%	2.44%	15.61%
	150.00%	-50.00%	7.50%	4.75%	21.80%
	175.00%	-75.00%	6.75%	7.94%	28.18%
	200.00%	-100.00%	6.00%	12.00%	34.65%

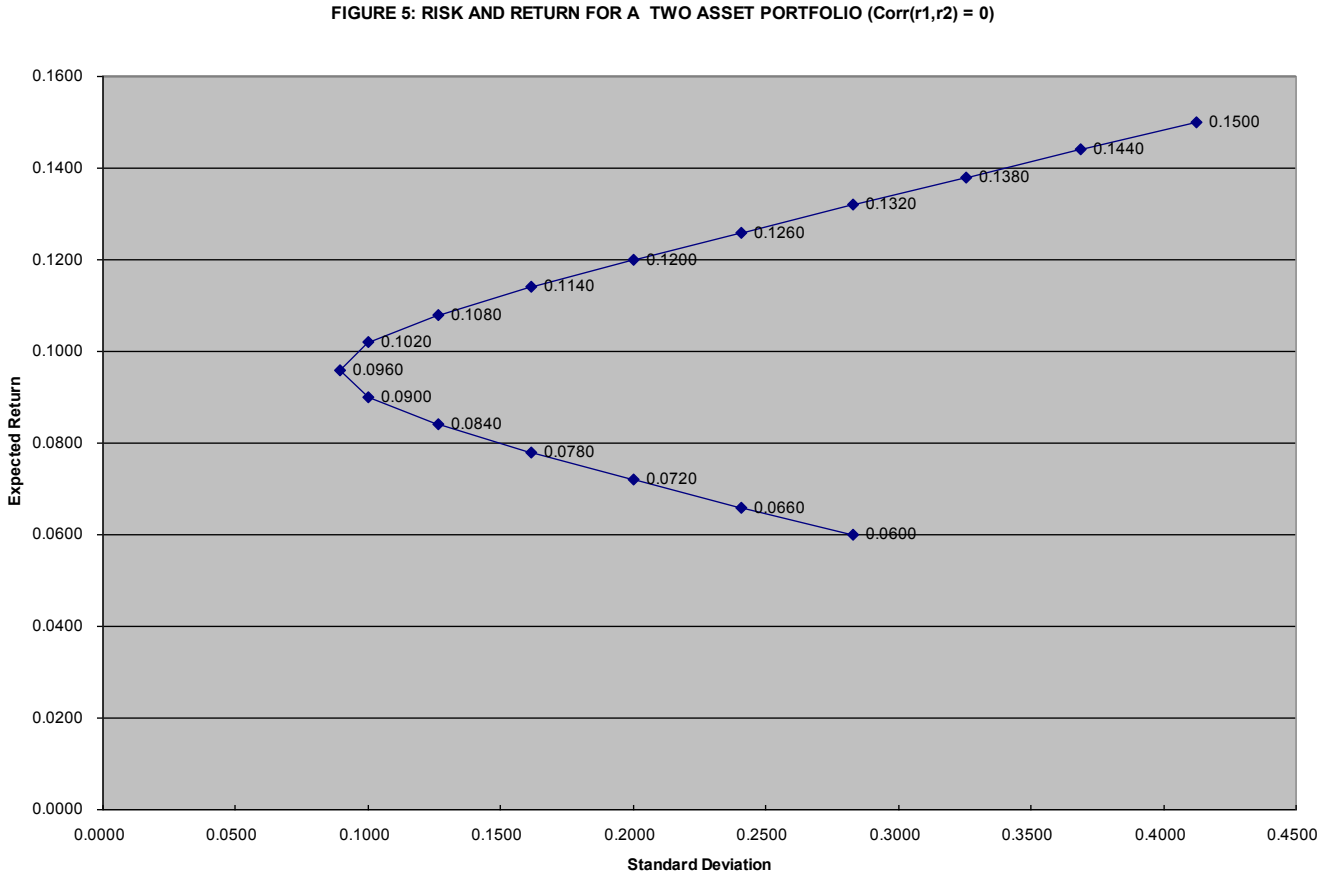
Portfolio Efficiency when $\rho = -.5$



Portfolio Efficiency when $\rho = 0$

Risks and Returns of Alternative Portfolio Strategies					
E(Ri)	9.00%	12.00%			
Var(Ri)	1.00%	4.00%			
Std(Ri)	10.00%	20.00%			
Cov(Ra,Rb)	0.0000				
Corr(Ra,Rb)	0.0000				
Minimum Var.	Xa	Xb	E(Rp)	Var(Rp)	Std(Rp)
Portfolio	80.00%	20.00%	9.60%	0.80%	8.95%
Other	**-100.00%**	**200.00%**	**15.00%**	**17.01%**	**41.24%**
Portfolios	**-80.00%**	**180.00%**	**14.40%**	**13.61%**	**36.89%**
	-60.00%	**160.00%**	**13.80%**	**10.60%**	**32.56%**
	-40.00%	**140.00%**	**13.20%**	**8.00%**	**28.29%**
	-20.00%	**120.00%**	**12.60%**	**5.80%**	**24.09%**
	0.00%	**100.00%**	**12.00%**	**4.00%**	**20.00%**
	20.00%	**80.00%**	**11.40%**	**2.60%**	**16.13%**
	40.00%	**60.00%**	**10.80%**	**1.60%**	**12.65%**
	60.00%	**40.00%**	**10.20%**	**1.00%**	**10.00%**
	80.00%	**20.00%**	**9.60%**	**0.80%**	**8.95%**
	100.00%	0.00%	9.00%	1.00%	10.00%
	120.00%	-20.00%	8.40%	1.60%	12.65%
	140.00%	-40.00%	7.80%	2.60%	16.13%
	160.00%	-60.00%	7.20%	4.00%	20.00%
	180.00%	-80.00%	6.60%	5.80%	24.09%
	200.00%	-100.00%	6.00%	8.00%	28.29%

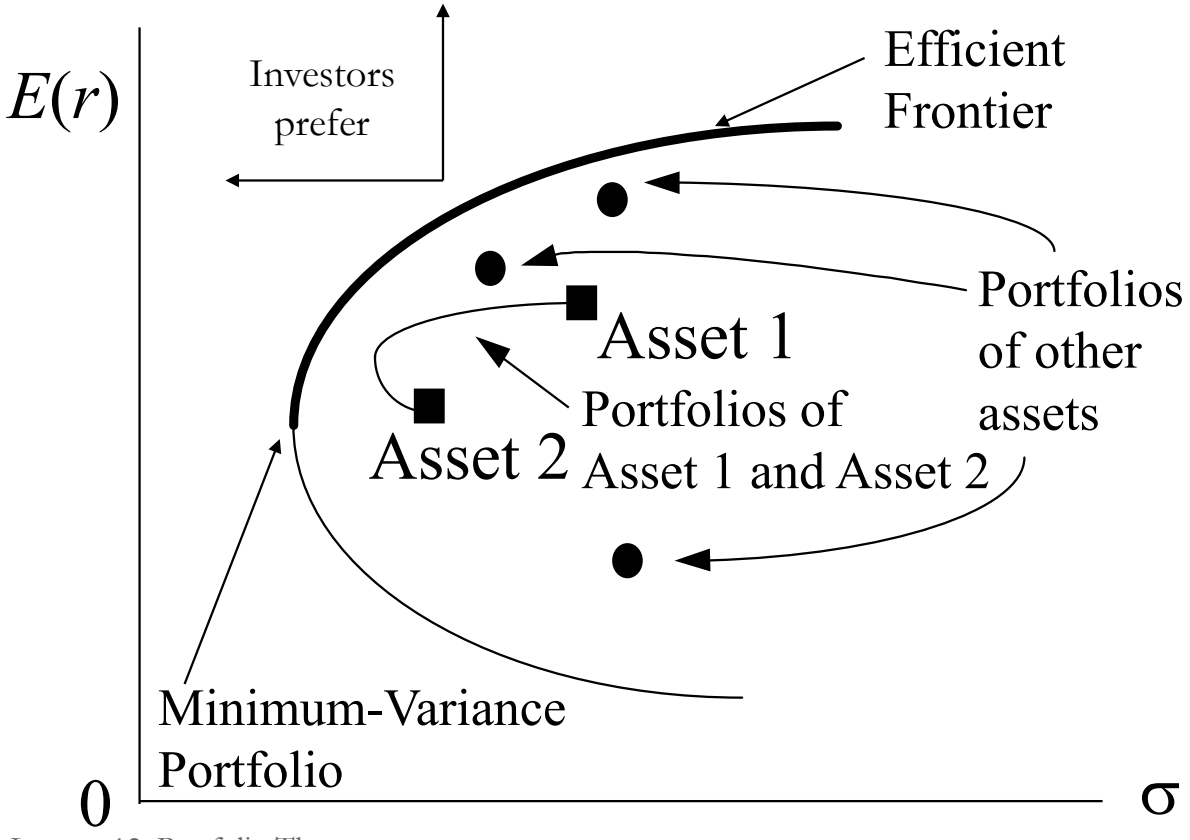
Portfolio Efficiency when $\rho = 0$



Efficient Portfolios with Multiple Assets

- Given $E(r_i)$, σ_i, and σ_{ij}, when there are n securities the investor:
  - determines which combinations of the n securities are mean variance efficient, and
  - selects a portfolio from the efficient set; this involves finding the portfolio that maximizes expected utility!
    - Irrespective of one's degree of risk aversion, investors agree upon and select from the same set of efficient portfolios.

Efficient Portfolios with Multiple Assets



1) $E(r_p) = w_1 E(r_1) + w_2 E(r_2)$

$E(r_1) = 12\%$, $E(r_2) = 8\%$

$\sigma_1 = 10\%$, $\sigma_2 = 4\%$, $\rho_{12} = -1$

2) $\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho_{12} \sigma_1 \sigma_2$

2. Min Risk combination of 1 and 2

$$\begin{cases} w_1 = \dfrac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} = \\ \uparrow \quad \uparrow \\ .01 \quad .04 \end{cases}$$